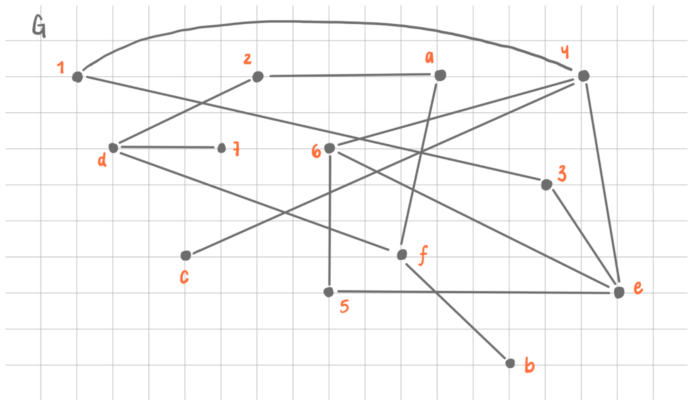
Grade 7/8 Math Circles

February 19th, 2024

Graph Theory: Isomorphisms - Problem Set

For the first four questions consider the graph G below:



1. For the graph G answer the following questions:
   (a) What is $V(G)$?
   (b) What is $E(G)$?
   (c) What are the neighbours and degree of each vertex in G?

Solution:

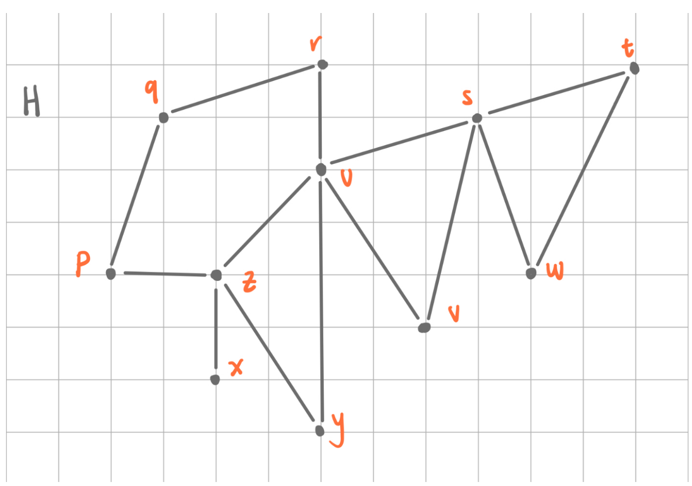
(a) $V(G) = \{1, 2, 3, 4, 5, 6, 7, a, b, c, d, e, f\}$

(b) $E(G) = \{\{1, 3\}, \{1, 4\}, \{2, a\}, \{2, d\}, \{3, e\}, \{4, 6\}, \{4, c\}, \{4, e\}, \{5, 6\}, \{5, e\}, \{6, e\}, \{7, d\}, \{a, f\}, \{b, f\}, \{d, f\}\}$

(c)

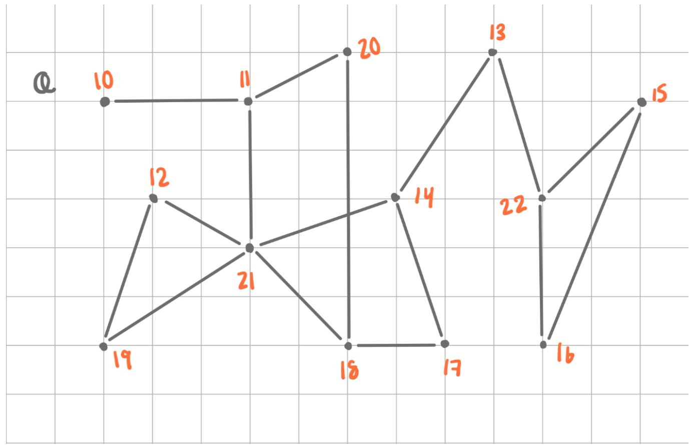
Vertex	Neighbours of Vertex	Degree of Vertex
1	3 and 4	2
2	a and d	2
3	1 and e	2
4	1, 6, c and e	4
5	6 and e	2
6	4, 5 and e	3
7	d	1
a	2 and f	2
b	f	1
c	4	1
d	2, 7 and f	3
e	3, 4, 5 and 6	4
f	a, b and d	3

2. Is G isomorphic to the graph H below? If yes provide an isomorphism if not explain why.



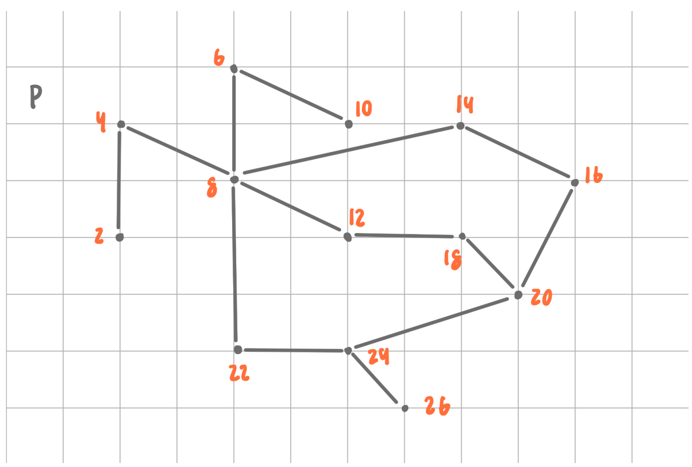
Solution: No, G is not isomorphic to H because G has 13 vertices while H only has 11 vertices, thus for any function $f : V(G) \Rightarrow V(H)$ that we build, f will not satisfy Criterion #2 of Graph Isomorphisms, since more than one vertex in G will have to be mapped to the same vertex in Q.

3. Is G isomorphic to the graph Q below? If yes provide an isomorphism if not explain why.



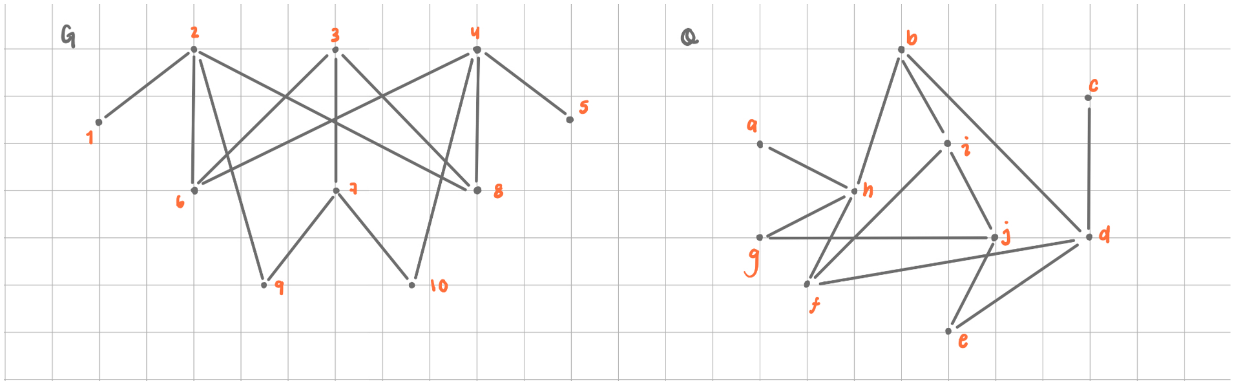
Solution: No, G is not isomorphic to Q because G has 3 vertices of degree 1, while Q only has 1, thus for any function $f : V(G) \Rightarrow V(H)$ that we build, f will not satisfy Criterion #1 of Graph Isomorphisms.

4. Is G isomorphic to the graph P below? If yes provide an isomorphism if not explain why.



Solution: No, G is not isomorphic to Q. Notice that in Q the vertex 8 has degree 5, but no vertex in G has degree greater than 4, thus for any function $f : V(G) \Rightarrow V(H)$ that we build, f will not satisfy Criterion #1 of Graph Isomorphisms.

For the next 4 Questions consider the **isomorphic** graphs G and Q below :



5. Is $f : V(G) \to V(Q)$ an isomorphism, where f is the following map? If it is an isomorphism then prove it, if not then explain why:

v	1	2	3	4	5	6	7	8	9	10
$f(v)$	a	b	c	d	e	f	g	h	i	j

Solution: To check that f is an isomorphism it suffices to show that f fulfills Criteria 1, 2 and 3 from the lesson. We'll first check that if u and v are adjacent vertices in G, then $f(u)$ and $f(v)$ are adjacent in Q. We'll do this by listing out the neighbours for each vertex u in G and seeing if they match up with the neighbours of $f(u)$ in Q:

u	$f(u)$	neighbours of u	relabelled neighbours of u	neighbours of $f(u)$
1	a	2	b	h

We can stop here since we've already found a problem with f. we know that $f(1) = a$, but from our chart above we see that the vertex 1 in G is a vertex of degree 2 in G with neighbours 3 and 4 but the vertex a in Q is a vertex of degree 1 with neighbour h, since the degrees and neighbours of 1 and a do not match up then we have that f has not satisfied Criterion #1 of Graph Isomorphisms, and so f is not an isomorphism.

6. Is $f : V(G) \to V(Q)$ an isomorphism, where f is the following map? If it is an isomorphism then prove it, if not then explain why:

v	1	2	3	4	5	6	7	8	9	10
$f(v)$	a	b	c	d	e	a	g	h	i	j

Solution: To check that f is an isomorphism it suffices to show that f fulfills Criteria 1, 2 and 3 from the lesson. Since $f(1) = a$ we know from part a that this already makes f fail Criterion #1 of Graph Isomorphisms. But we can also notice something crucial by looking at our input-output chart! In the outputs we can see that $f(1) = a$ and $f(6) = a$ but we know that vertex 1 $\neq$ vertex 6, thus f fails Criterion #2 of Graph Isomorphisms as well, and so f is not an isomorphism. But there is one last thing we could've noticed by looking at our input-output chart, in the outputs we can see that the vertex f does not show up and so f fails Criterion #3 of Graph Isomorphisms as well. Thus clearly since f fails all of Criteria 1, 2 and 3 from the lesson then f is not an isomorphism.

7. Is $f : V(G) \to V(Q)$ an isomorphism, where f is the following map? If it is an isomorphism then prove it, if not then explain why:

v	1	2	3	4	5	6	7	8	9	10
$f(v)$	a	h	i	d	c	b	j	f	g	e

Solution: To check that f is an isomorphism it suffices to show that f fulfills Criteria 1, 2 and 3 from the lesson. We'll first check that if u and v are adjacent vertices in G, then $f(u)$ and $f(v)$ are adjacent in Q. We'll do this by listing out the neighbours for each vertex u in G and seeing if they match up with the neighbours of $f(u)$ in Q:

u	$f(u)$	neighbours of u	relabelled neighbours of u	neighbours of $f(u)$
1	a	2	h	h
2	h	6, 8 and 9	b, f, and g	b, f, and g
3	i	6, 7 and 8	b, j and f	b, j and f
4	d	5, 6 and 8	c, b and f	c, b and f
5	c	4	d	d
6	b	2, 3 and 4	h, i and d	h, i and d
7	j	3, 9 and 10	i, g and e	i, g and e
8	f	2, 3 and 4	h, i and d	h, i and d
9	g	2 and 7	h and j	h and j
10	e	4 and 7	d and j	d and j

Now notice the last two columns of the chart above show that if u and v are adjacent vertices in G, then $f(u)$ and $f(v)$ are adjacent in Q since they match up exactly. Thus $f : V(G) \to V(Q)$ fulfills Criterion #1 of Graph Isomorphisms. Now looking at our input-output table we can see that each vertex in Q appears exactly once- this immediately tells us that $f : V(G) \to V(Q)$ fulfills Criterion #2 and #3 of Graph Isomorphisms! Therefore

f fulfills Criteria 1, 2 and 3 from the lesson and so f is an isomorphism.

8. Is $f : V(G) \to V(Q)$ an isomorphism, where f is the following map? If it is an isomorphism then prove it, if not then explain why:

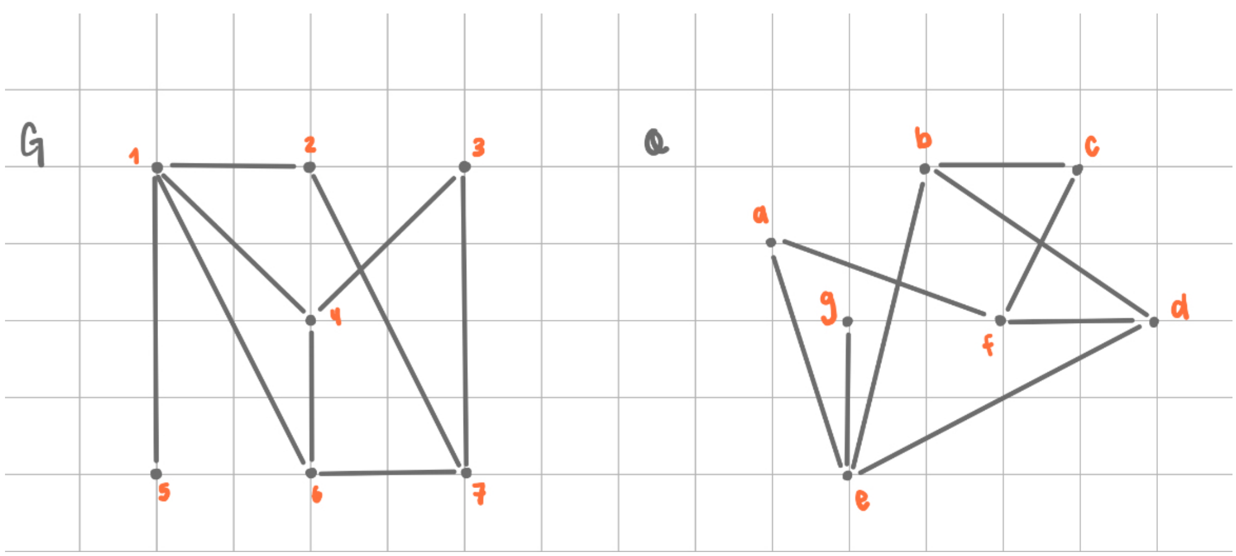
v	1	2	3	4	5	6	7	8	9	10
$f(v)$	b	a	f	i	c	j	g	h	d	e

Solution: To check that f is an isomorphism it suffices to show that f fulfills Criteria 1, 2 and 3 from the lesson. We'll first check that if u and v are adjacent vertices in G, then $f(u)$ and $f(v)$ are adjacent in Q. We'll do this by listing out the neighbours for each vertex u in G and seeing if they match up with the neighbours of $f(u)$ in Q:

u	$f(u)$	neighbours of u	relabelled neighbours of u	neighbours of $f(u)$
1	b	2	a	h

We can stop here since we've already found a problem with f. we know that $f(1) = b$ and since 2 is the neighbour of 1 in G we need $f(2) = a$ to be a neighbour to $f(1) = b$ in Q, but we know that a and b are not neighbours in Q and so f fail Criterion #1 of Graph Isomorphisms, telling us that f is not an isomorphism.

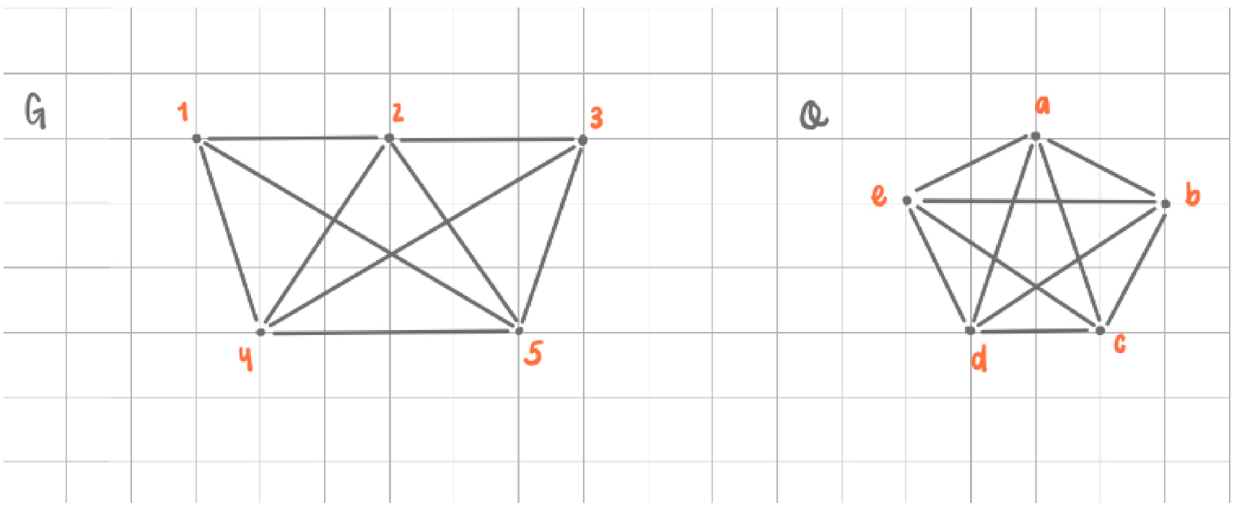
9. Are the following two graphs G and Q isomorphic? If yes provide an isomorphism, if not then state why.



Solution: Yes, the graphs G and Q are isomorphic! Consider the isomorphism $f : V(G) \to V(Q)$ given in the following input-output table;

u	1	2	3	4	5	6	7
$f(u)$	e	a	c	b	g	d	f

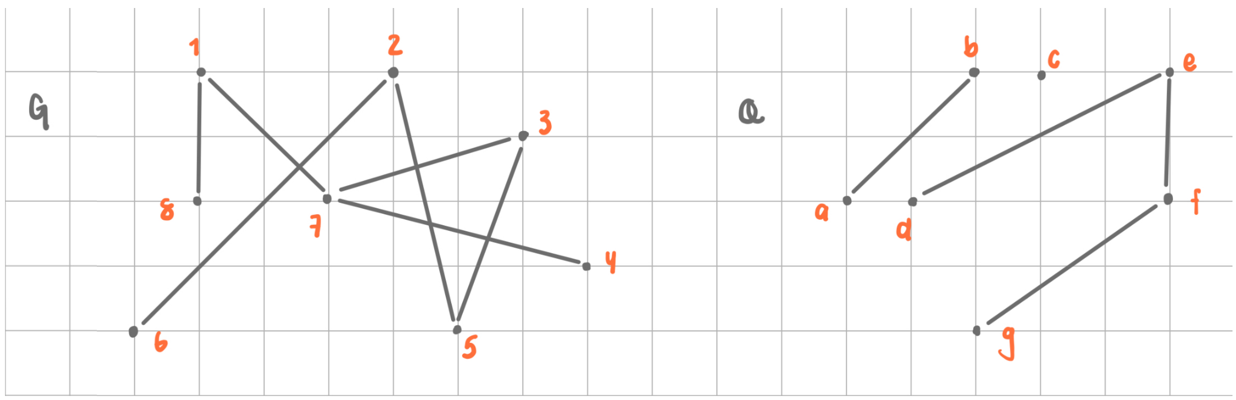
10. * The following two graphs G and Q not isomorphic. With one change (adding/removing on edge or one vertex) how could you make these two graphs isomorphic? Prove that after the change the graphs are isomorphic.



Solution: The change is to add an edge between vertices 1 and 3 in G. Then the function $f : V(G) \to V(Q)$ is an isomorphism, where f is given by: the following input-output table;

u	1	2	3	4	5
$f(u)$	e	a	b	d	c

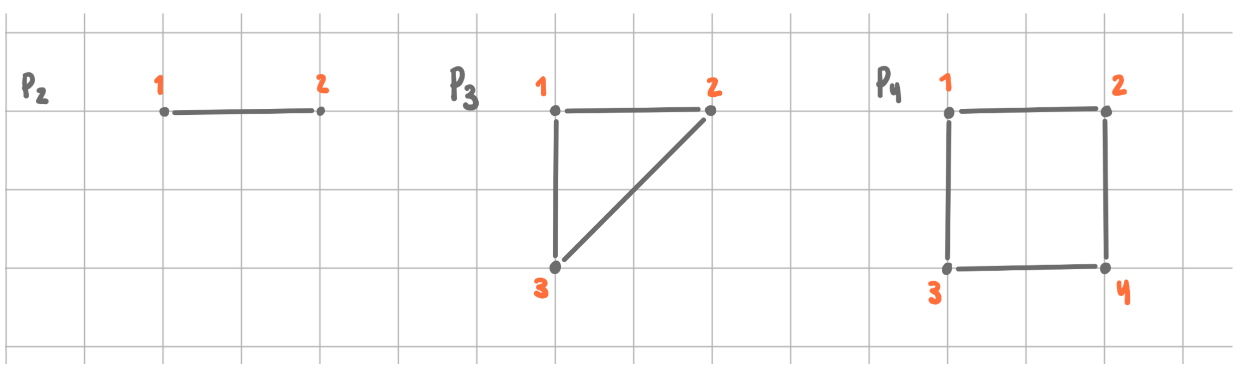
11. * The following two graphs G and Q not isomorphic. With one change (adding/removing on edge or one vertex) how could you make these two graphs isomorphic? Prove that after the change the graphs are isomorphic.



Solution: The change is to remove vertex 7 from G. Then the function $f : V(G) \to V(Q)$ is an isomorphism, where f is given by: the following input-output table;

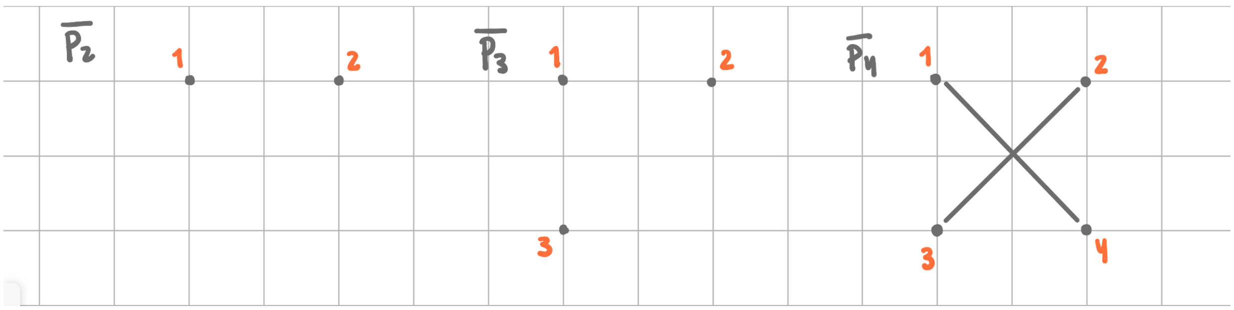
u	1	2	3	4	5	6	8
$f(u)$	a	e	g	c	f	d	b

12. *** Below are the graphs P_2, P_3, and P_4 from the family of Polygon Graphs, the polygon graph P_n is simply the regular polygon with n sides (P_3 is a triangle, P_4 is a rectangle, P_5 is a pentagon etc):



(a) Draw and label the graphs P_5, P_6, and P_7.

(b) We define the complement of a graph G as $\bar{G}$ to be a graph with the same vertex set as G, but has an edge set in which any edge that is not in G is an edge of $\bar{G}$. Below are the graphs of $\bar{P}_2$, $\bar{P}_3$, and $\bar{P}_4$. Draw and label the graphs of $\bar{P}_5$, $\bar{P}_6$, and $\bar{P}_7$.

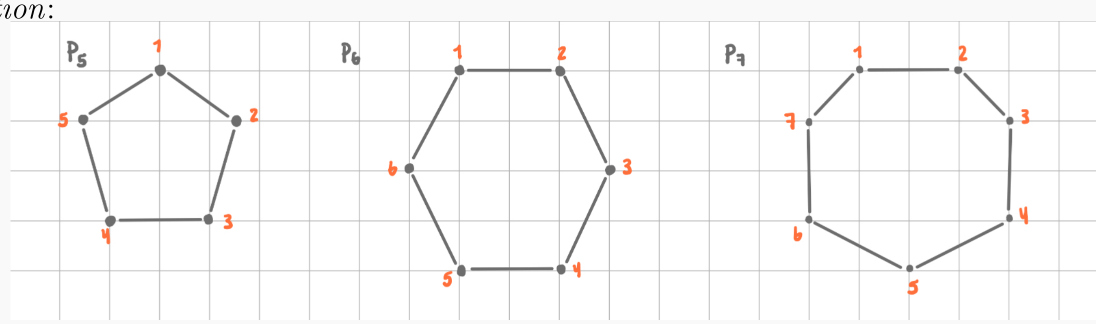


(c) Which of P_2, P_3, P_4, P_5, P_6, and P_7 are isomorphic to their complement, state which one(s) are isomorphic and provide and isomorphism.

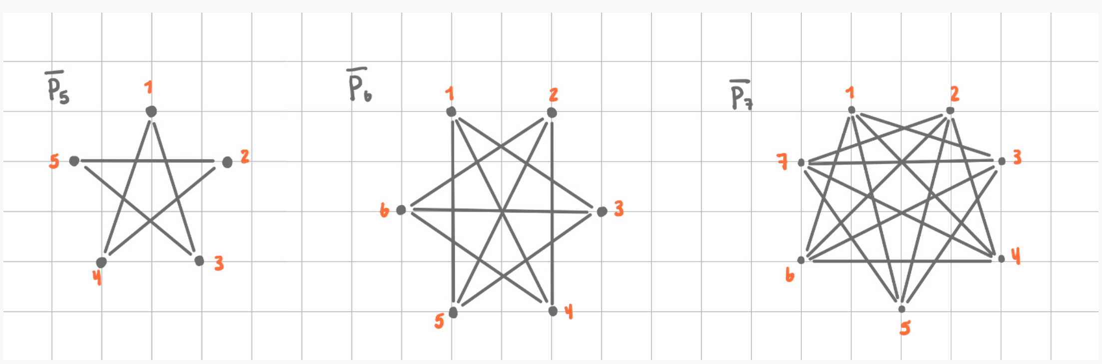
(d) Besides the isomorphic graph(s) you found in part c is there any other graph in the Polygon Graph family which will be isomorphic to its complement? Explain your reasoning.

Solution:

(a)



(b)



(c) Only P_5 is isomorphic to its complement- the isomorphism is as follows:

u	1	2	3	4	5
$f(u)$	1	3	5	2	4

(d) Only P_5 is isomorphic to its complement, notice that in each graph P_n all vertices have degree equal to 2, so for P_n to be isomorphic to its complement we require that all vertices in $\bar{P}_n$ also have degree equal to 2. If all vertices in $\bar{P}_n$ have degree equal to 2 then this means that for a given vertex u in P_n, u is not adjacent to exactly two vertices, and we also know that all vertices in P_n have degree equal to 2 so in P_n there are two vertices adjacent to u, two vertices not adjacent to u and u its self, this tells us that P_n has exactly 5 vertices. Therefore only P_5 is isomorphic to its complement.